Notes for MA591U, Spring 2001 (Symbolic Computation)

Liouville's Theorem (Applications)

Consider $\int fe^g$ where $f, g \in \mathbb{C}(x)$.

LEMMA: If $g \in \mathbb{C}(x)\backslash\mathbb{C}$, then e^g is not algebraic over $\mathbb{C}(x)$; that is, $\neg\exists y$ that is algebraic over $\mathbb{C}(x)$ such that $y' = g'y$.

PROOF:

By way of contradiction, we will assume there *is* some such y satisfying $y' = g'y$, and pick it. Let $y^n + a_{n-1}y^{n-1} + \cdots + a_0 = 0$ where the polynomial is irreducible over $\mathbb{C}(x)$ and the $a_i \in \mathbb{C}(x)$. Differentiate this equation:

$$\begin{aligned} 0 &= ny^{n-1}y' + a'_{n-1}y^{n-1} + a_{n-1}(n-1)y^{n-1}y' + \cdots \\ &= ng'y^n + (a'_{n-1} + (n-1)a_{n-1}g')y^{n-1} + \cdots \end{aligned}$$

Since the previous polynomial was irreducible, this new polynomial must be a multiple of that one:

$$ng'y^n + (a'_{n-1} + (n-1)a_{n-1}g')y^{n-1} + \cdots + a'_0 = ng'(y^n + a_{n-1}y^{n-1} + \cdots + a_0)$$

so $ng'a_0 = a'_0$; that is,

$$\frac{a'_0}{a_0} = ng'.$$

Now, can one *have* such a relationship with a_0 in $\mathbb{C}(x)$? The answer is *NO*. Write

$$a_0 = \prod_i (x - \alpha_i)^{n_i} \ \exists \alpha_i \in \mathbb{C}, n_i \in \mathbb{Z}.$$

So

$$\frac{a'_0}{a_0} = \sum_i \frac{n_i}{x - \alpha_i}.$$

Write

$$g = h + \sum_i \sum_j \frac{d_{ij}}{(x - \beta_i)^j}$$

and we have

$$g' = h' + \sum_i \sum_j \frac{-jd_{ij}}{(x - \beta_i)^{j+1}}.$$

Consider again the relation $\frac{a_0'}{a_0} = ng'$. On the left, we have only terms of the form $\frac{c}{x-\alpha}$; on the right, we have no such terms, unless $g \in \mathbb{C}$. This contradicts the assumption.

PROPOSITION: Let $f, g \in \mathbb{C}(x)$ with $f \neq 0$ and $g \notin \mathbb{C}$. Then fe^g has an elementary antiderivative if, and only if, $\exists a \in \mathbb{C}(x)$ such that $f = a' + ag'$, in which case $\int fe^g = ae^g$.
PROOF:

One direction is easy: if such an a exists then $\int fe^g = \int (a' + ag')e^g = ae^g$.

Let $t = e^g$ and $\mathbb{F} = \mathbb{C}(x)$. Note that t is not algebraic over $\mathbb{F}$ by the lemma above. So $fe^g = ft \in \mathbb{F}[t]$. Suppose ft has an elementary integral. Then Liouville's Theorem implies that

$$ft = v' + \sum c_i \frac{u_i'}{u_i}$$

with $v_i, u_i \in \mathbb{F}(t)$ and c_i constant. We can assume that $u_i \in \mathbb{F}[t]$, and that they are distinct, monic, and irreducible (use the same arguments as in the prequel). In the proof of Liouville's Theorem, case 3, we gave an argument that we can use here to show that all the $u_i \in \mathbb{F}$, except possibly one, say, $u_1 = t$. So

$$\begin{aligned} ft &= v' + c_1 \frac{t'}{t} + \sum c_i \frac{u_i'}{u_i} \\ &= v' + c_1 g' + \sum c_i \frac{u_i'}{u_i} \end{aligned}$$

Observe that $c_1 g' + \sum c_i \frac{u_i'}{u_i} \in \mathbb{F}$. Using the technical lemma from a few days back, one then shows that $\exists b_o, b_1 \in \mathbb{F}$ such that $v = b_0 + b_1 t$. Set $h = c_1 g' + \sum c_i \frac{u_i'}{u_i}$ and we have

$$\begin{aligned} ft &= (b_0 + b_1 t)' + h \\ &= b_0' + (b_1' + g' b_1)t + h \end{aligned}$$

Comparing the coefficients of t, we have $f = b_1' + g' b_1$. Let $a = b_1$ and we are done.

EXAMPLE: $\int e^{x^2}$ has no elementary expression.

Using the notation of the proposition, $f = 1$ and $g = x^2$. We want to know if there exists some $a \in \mathbb{C}(x)$ with $a' + 2xa = 1$. Write

$$a = h(x) + \sum_i \sum_j \frac{d_{ij}}{(x - \alpha_i)^j}$$

so

$$a' = h'(x) + \sum_i \sum_j \frac{-j d_{ij}}{(x - \alpha_i)^{j+1}}.$$

Fix $\alpha = \alpha_i$ and let n be the largest power of $x - \alpha$ in the denominator of a. Then $(x-\alpha)^{n+1}$ appears in the denominator of a' but not in the denominator of a, so that we cannot cancel and thereby obtain $a' + 2xa = 1$. Hence $a = h(x) \in \mathbb{C}(x)$. Then we have $x^{\deg h}$ in a but not in a', and again we cannot cancel to obtain 1. So no such a exists, and hence $\int e^{x^2}$ has no elementary expression.

REMARK: Given $f, g \in \mathbb{Q}(x)$, one can always decide if there is an $a \in \mathbb{C}(x)$ with $a' + ag' = f$, and if there is, one can find it. We will come back to this question later.