Notes for MA591U, Spring 2001 (Symbolic Computation)

The Risch Algorithm

Risch showed in 1970 that "Given an elementary function f, one can decide if $\int f$ is also elementary, and if so, find $\int f$."

The complete proof never appeared in print. The parts that did, appeared in:

- *AMS Transactions* 139 May 1969 p. 167-189
- *Bulletins of the AMS* 76 1970 p. 605-608
- preprints

There are some problems with the question we have posed. To begin with, what do we mean by, "given an elementary function"? We will consider several possible interpretations.

First: we can write the function down. But this is not enough.

Second: we can write down an elementary extension containing f. For example, let $f(x) = \sqrt{x}e^x = e^{x+\frac{1}{2}\ln x}$. Observe that $f \in \mathbb{Q}(x, \sqrt{x}, e^x)$, but also $f \in \mathbb{Q}(x, \ln x, e^{x+\frac{1}{2}\ln x})$. We also have to make *this* answer more precise. We can demosntrate this problem as follows: consider $\mathbb{Q}(x, \ln x, e^{\ln x})$. What does $e^{\ln x}$ mean? It stands for a function whose logarithmic derivative is

$$\frac{(e^{\ln x})'}{e^{\ln x}} = \frac{1}{x}.$$

But we could use, for $\ln x$, the function as defined on $\mathbb{R}_{>0}$, or we could use $\ln x + 2n\pi i$ for any $n \in \mathbb{C}$. Which one do we mean to use? If we use the usual $\ln x$, we get $\mathbb{Q}(x, e^x, \ln e^x) = \mathbb{Q}(x, e^x)$; if we use $\ln x + 2\pi i$ we obtain $\mathbb{Q}(x, e^x, 2\pi i)$.

Risch showed in his 1969 paper that if one does not do something to avoid the ambiguity, the question "Does f have an elementary integral?" is undecidable.

Definition: A *recursive description of an elementary extension* (RDEE) $\mathbb{E}$ of $\mathbb{Q}(x)$ is the following:

(i) a set of elements $S = \{\alpha_1, \ldots, \alpha_m\}$ such that the constants of $\mathbb{E}$ are $\mathbb{Q}(\alpha_1, \ldots, \alpha_m)$ where $\alpha_1, \ldots, \alpha_{m-1}$ are algebraically independent, and α_m is algebraic over $\mathbb{Q}(\alpha_1, \ldots \alpha_{m-1})$ with a given minimal polynomial.

(ii) a set of elements $T = \{t_1, \ldots, t_n\}$ such that we know in advance which of the following holds: let $\mathbb{E}_i = \mathbb{Q}(\alpha_1, \ldots, \alpha_m)(x)(t_1, \ldots, t_{i-1})$ where $x' = 1$ and

(a) t_i is algebraic over $\mathbb{E}_i$ and we have an explicit description of its minimal polynomial, or

(b) t_i is not algebraic over $\mathbb{E}_i$ and we have an element $u_i \in \mathbb{E}_i$ such that $t_i' = u_i'/u_i$, or

(c) t_i is not algebraic over $\mathbb{E}_i$ and we gave ab element $u_i \in \mathbb{E}_i$ such that $t_i'/t_i = u_i'$.

REMARKS:
(1) Given a RDEE $\mathbb{E}$, we can effectively perform the operations of $+, -, \cdot, \div$, and factoring of polynomials over $\mathbb{E}$.
(2) An elementary extension $\mathbb{E}$ over $\mathbb{Q}(x)$ can have several RDEE's. For example, if $\mathbb{E} = \mathbb{Q}(x, e^x)$, then

(i) $S = \emptyset$, $T = \{t_1 = e^x\}$ so case (ii,b) above holds for $u_1 = x$.
(ii) $S = \emptyset$, $T = \{t_1 = e^{2x}, t_2 = e^x\}$ and then $t_1'/t_1 = (2x)'$ so case (ii)(c) above holds, with t_2 algebraic over $\mathbb{E}_2 = \mathbb{Q}(x)(e^{2x})$, since $t_2^2 - t_1 = 0$.

(3) An elementary function can belong to several different elementary extensions. For example,

$$\begin{aligned} \sqrt{x}e^x &= e^{x+\frac{1}{2}\ln x} = \sqrt{xe^{2x}} \text{ and so} \\ \sqrt{x}e^x &\in \mathbb{Q}(x)(\ln x, e^{x+\frac{1}{2}\ln x}) \doteq \mathbb{E}_1 \\ &\in \mathbb{Q}(x)(\sqrt{x}, e^x) \doteq \mathbb{E}_2 \\ &\in \mathbb{Q}(x)(e^{2x}, \sqrt{xe^{2x}}) \doteq \mathbb{E}_3 = \mathbb{E}_1 \cap \mathbb{E}_2 \end{aligned}$$

(4) Risch gave a procedure to do the following: given an expression for an elementary function, construct some RDEE ofa field $\mathbb{E}$ containing an element corresponding to the expression.

How does RDEE help us get around the difficulty of how to simplify $\ln e^x$? Consider again $\mathbb{Q}(x, e^x, \ln e^x)$. Set $t_1 = e^x$, $u_1 = x$, so $t_1'/t_1 = u_1'$. We can say that t_2 is algebraic over E_1: $t_2 - x = 0$.

Or again, let $S = \{\pi i\}$, $T = \{t_1 = e^x, t_2 = \ln e^x\}$ with the constants being $\mathbb{Q}(\pi i)$. We have $t_1'/t_1 = x'$ and again t_2 is algebraic over $\mathbb{E}_1$ with $t_2 - (x + 2\pi i) = 0$.

In other words, going by RDEE forces us to make explicit which simplification to use.

DEFINITION: A *transcendental RDEE* of an elementary extension $\mathbb{E}$ of $\mathbb{Q}(x)$ is an RDEE where case (ii,a) above does *not* occur; that is, there are no algebraic extensions.

EXAMPLES:
(1) $\mathbb{E}_1$ above is a TRDEE while $\mathbb{E}_2$ and $\mathbb{E}_3$ are not.
(2) $\mathbb{Q}(x, \sqrt{x})$ has no TRDEE.

NOTATION: We write $\overline{k}$ for the algebraic closure of a field k, and $\overline{k} \cdot \mathbb{E}$ for the smallest field containing $\overline{k}$ and $\mathbb{E}$.

THEOREM: *(Risch, 1969 TAMS)*

Let $\mathbb{E}$ be an elementary extension of $\mathbb{Q}(x)$ with a TRDEE, and let k be its field of constants. For every $f \in \mathbb{E}$, one can determine in a finite number of steps if $\int f$ is elementary over $\mathbb{E}$. If so, one can find $v_0 \in \mathbb{E}$, $v_i \in \overline{k} \cdot \mathbb{E}$, and $c_i \in \overline{k}$ so that

$$f = v_0' + \sum_i c_i \frac{v_i'}{v_i}.$$

The algorithm has an efficient implementation (polynomial time). Risch outlined how to extend the algorithm to RDEE's in 1970. This, however, is much, much harder.

OPEN PROBLEMS:
(1) Given a RDEE $\mathbb{E}$, determine if there is a TRDEE.
(2) "Given" an elementary function, find "the best" RDEE $\mathbb{E}$ containing this function.

In "An Extension of Liouville's Theorem on Integration in Finite Terms" (*SIAM Journal of Computing*, vol. 14 no. 4, 1985), Singer, Caviness, and Saunders showed that there is an algorithm for problem (2) for certain "special" RDEE's.

EXTENSIONS OF LIOUVILLE'S THEOREM AND RISCH'S ALGORITHM
(1) Allow $\ln$, $\exp$, algebraics, and the error function

$$\text{erf}(x) \doteq \int e^{-x^2} dx.$$

Singer, Caviness, and Saunders give a Liouville-type theorem in the publication listed above, along with a partial algorithm. G. Cherry generalized this work in *SIAM Journal of Computing* vol. 15 no. 1 (1986) and *The Journal of Symbolic Computation* vol. 1 no. 3 (1985). P. Knowles also extended it in his Ph.D. thesis for North Carolina State University (1986), appearing in *The Journal of Symbolic Computation* in 1991.
(2) Allow $\ln$, $\exp$, algebraics, and

$$\text{dilog}(x) \doteq \int \frac{\ln x}{x+1} dx.$$

J. Baddoura gave a result in this regard in *Proc. Comp. and Math*, edited by Kaltofen and Watt, Springer (1989).

EXAMPLE: What does Risch do with $\int \frac{dx}{\sin x + 2}$? In an appendix to Kaltofen's 2000 paper "Challenges of Symbolic Computation" Corless and Jeffrey show that Maple (for example) produces a discontinuous antiderivative, in spite of the fact that the area under the curve in question has no problem points. So the Risch algorithm needs modification, or perhaps replacement, as we would like to obtain an antiderivative that evaluates correctly, or that at least evaluates correctly on the largest possible continuous domian.