Notes for MA591U, Spring 2001
(Symbolic Computation)

Liouville's Theorem (Introduction)

Let $f/g \in \mathbb{Q}(x)$. We know that

$$\exists c_i \in \mathbb{C}, \exists u_i \in \mathbb{C}(x) \left(\int \frac{f}{g} = u_0 + \sum_{i=1}^{t} c_i \ln u_i \right).$$

We want to know when one can do this "in general". That is, when can we integrate an "elementary function" in terms of elementary functions? Liouville gave a necessary and sufficient condition, and showed that $\int e^{x^2}$ is not elementary.

In 1969, Risch turned this criterion into an algorithm.

"DEFINITION": An **elementary function** is one that we can build up from rational functions using $\exp$, $\ln$, and arbitrary algebraic roots.

EXAMPLES:

(1) $\sqrt{\exp(x) + \ln(x+1)}$

(2) Any function $y(x)$ that satisfies $[y(x)]^5 + xy(x) + e^x = 0$.

This leads us to ask: what do we mean by "build up"? Here we need to define a few things:

(1) a differential field,

(2) elementary extensions of differential fields.

DEFINITION: Let R be a (commutative) ring. A map $D : R \to R$ is a **derivation** if:

(a) $D(a+b) = D(a) + D(b)$

(b) $D(ab) = D(a) \cdot b + a \cdot D(b)$

A **differential ring (field)** is a ring (field) with a derivation.

EXAMPLES:

$(\mathbb{C}(x), D = \frac{d}{dx})$

$(\mathbb{C}[x], D = \frac{d}{dx})$

$(\mathbb{C}(x), D = x\frac{d}{sx})$

If (R, D) is a differential ring, so is (R, rD) for any $r \in R$.

NOTE: For $(\mathbb{C}(x), x\frac{d}{dx})$, we have $x\frac{d}{dx}(x) = x$ so $(\mathbb{C}(x), x\frac{d}{dx}) \cong (\mathbb{C}(e^x), \frac{d}{dx})$.

NOTATION: We will often write r' for $D(r)$.

LEMMA: Let k be a differential field , with $a, b \in k$ and $b \neq 0$. Then

(i) $1' = 0$

(ii) $(b^{-1})' = -b'(b^{-1})^2$

(iii) $(ab^{-1})' = (a'b - ba')(b^{-1})^2$

(iv) $(a^n)' = na^{n-1} \cdot a'$

(v) $\{c \in k : c' = 0\}$ is a field, which we call the **field of constants**.

PROOF:

(i) $1 = 1 \cdot 1$ so

$$D(1) = D(1 \cdot 1) = D(1) \cdot 1 + 1 \cdot D(1) = 2D(1)$$

and $[D(1) = 2D(1)] \Rightarrow [D(1) = 0]$ (since a field has no zero divisors).

(ii) $1 = b \cdot b^{-1}$ so

$$0 = 1' = (b \cdot b^{-1})' = b' \cdot b^{-1} + b \cdot (b^{-1})'.$$

Thus $-b'b^{-1} = b \cdot (b^{-1})'$, and we see immediately that $-b'(b^{-1})^2 = (b^{-1})'$.

(iii) follows from (ii).

(iv) follows by induction on n.

(v) follows from the definition of derivative, and (ii).

LEMMA: *(Logarithmic Derivative Identity)*

$$\frac{(a_1^{\nu_1} \cdots a_n^{\nu_n})'}{a_1^{\nu_1} \cdots a_n^{\nu_n}} = \nu_1 \frac{a_1'}{a_1} + \cdots + \nu_n \frac{a_n'}{a_n}.$$

PROOF:

The product rule.

The name of the above comes from the fact that, given $r \in R$, we define the logarithmic derivative of r as:

$$\ell D(r) = \frac{r'}{r}.$$

The lemma merely points out that $\ell D(rs) = \ell D(r) + \ell D(s)$.