Notes for MA591U, Spring 2001 (Symbolic Computation)

Linear Differential Equations (Rational Solutions of LDEs)

Let $\mathbb{Q} \subset C \subset \widetilde{C}$ with $\widetilde{C}$ algebraic over C. Let $L(y) = y^{(n)} + a_{n-1}(x)y^{(n-1)} + \cdots + a_0(x)y$ with the $a_i(x) \in C(x)$.

PROPOSITION: (Bronstein, in the *Journal of Symbolic Computation* vol. 13 no. 4 1992)
1. Let $b \in C(x)$. If $L(y) = b(x)$ has a solution that is algebraic over $\widetilde{C}(x)$, then it has one that is in $C(x)$.
2. Let V be the C-vector space of solutions of $L(y) = 0$ in $C(x)$, and let $\widetilde{V}$ be the $\widetilde{C}$-vector space of solutions of $L(y) = 0$ in $\widetilde{C}(x)$. Then any C-basis of V is a $\widetilde{C}$-basis of $\widetilde{V}$. So $\dim_C V = \dim_{\widetilde{C}} \widetilde{V}$.
3. One can decide if $L(y) = b(x)$ has a solution in $C(x)$, and find a basis for the solution space of $L(y) = 0$ in $C(x)$.

We omit the proof, but present an illustration of the third case for two examples:

$$y' + fy = g \text{ and } y'' + fy' + gy = 0$$

for any $f, g \in \mathbb{Q}(x)$.

NOTE: To decide if $\int Fe^G$ is elementary, we need to find rational solutions of $y' + G'y = F$.

1. We begin with the first example. Let $f, y \in \mathbb{Q}(x)$ such that $y' + fy = g$. We want to decide if this has a solution in $\mathbb{Q}(x)$, and to find one if so. There are two steps.

(i) Find a denominator D for any possible solution.

(ii) Let $y = Y/D$, substitute, clear the denominators, and decide if the new equation $AY' + BY = C$, with $A, B, C \in \mathbb{Q}[x]$, has a polynomial solution.

(iii) If so, solve the resulting linear system.

(1.i) Let $y = Y/D$ be a solution, with $Y = \prod_i p_i^{n_i}$ where p_i is irreducible in $\mathbb{Q}[x]$ and $n_i \in \mathbb{N}$. We want to determine which p_i can occur, and bound the n_i. Fix some $p_i = p$. Then we can write

$$f = \frac{a}{p^\alpha} + \widetilde{f} \text{ and } g = \frac{b}{p^\beta} + \widetilde{g}$$

where $\widetilde{f}$ and $\widetilde{g}$ are partial fraction decompositions whose denominators have smaller powers of p. We know the values of $\alpha, \beta, a,$ and b (with $\deg a, \deg b < \deg p$).

Let $y = q/p^n + \widetilde{y}$ and plug it in. Consider the equation $y' + fy = g$. We have

$$\left(\frac{-np'q}{p^{n+1}} + \cdots\right) + \left(\frac{qa}{p^{n+\alpha}} + \cdots\right) = \frac{b}{p^\beta} + \cdots$$

where we have exhibited the "leading terms".

In order to have the two sides equal, we face three possibilities here in terms of cancellation.

(a) $\alpha > 1 : \ (n+\alpha = \beta > n+1) \Rightarrow (n = \beta - \alpha)$
(b) $\alpha = 0 : \ (n+1 = \beta > n+\alpha) \Rightarrow (n = \beta - 1)$
(c) $\alpha = 1 : \ n+1 = n+\alpha > \beta$

If these numbers are integers, then there is a rational solution; proceed as follows. Otherwise, there is no rational solution. This last possibility will also bound n for us since it means that we have

$$\begin{aligned} -np'q + qa &= 0 \\ (-np' + a)q &= 0 \end{aligned}$$

Since $q \neq 0$, it must be that $-np' + a = 0$, and thus $n = a/p'$.

Hence if $p|D$, we see that p must occur in the denominator of f or g, and the power of p in D is one of $\beta - \alpha$, $\beta - 1$, or a/p'.

EXAMPLE: Suppose $y' + (10/x)y = 0$. We set $f = 10/x$ and $g = 0$. Let $p = x$. Observe that $a = 10$ and $b = 0$. We get $a/p' = 10/1 = 10$, and sure enough $y = 1/x^{10}$ is a solution to the LDE.

(1.ii) Now write $D = \prod_i p_i^{n_i}$, where the p_i consist of the irreducible factors of the denominators of f and g, and n_i is the largest integer of $\{\beta_i - \alpha_i, \beta_i - 1, a_i/p_i'\}$. Let $y = Y/D$ where Y is a new variable. Then we have the equation

$$\left(\frac{Y}{D}\right)' + \frac{F}{D} \cdot \frac{Y}{D} = \frac{G}{D}$$

where $f = F/D$ and $g = G/D$. Then

$$\begin{aligned} \frac{Y'}{D} - \frac{YD'}{D^2} + \frac{FY}{D^2} &= \frac{G}{D} \\ Y'D - YD' + FY &= GD \\ Y'D - Y(D' + F) &= GD \end{aligned}$$

which we can write as $AY' + BY = C$ with $A, B, C \in \mathbb{Q}[x]$. We want polynomial solutions to this new equation.

Write $A = ax^\alpha + \cdots$, $B = bx^\beta + \cdots$, $C = cx^\gamma + \cdots$, and $Y = dx^n + \cdots$, where we have exhibited the leading terms. First we want to bound n. Plug in and observe that

$$(nadx^{\alpha+n-1} + \cdots) + (bdx^{\beta+n} + \cdots) = cx^\gamma + \cdots$$

and again we have three cases.

(a) $(\alpha + n - 1 = \gamma) \Rightarrow (n = \gamma - \alpha + 1)$

(b) $(\beta + n = \gamma) \Rightarrow (n = \gamma - \beta)$

(c) $(\alpha + n - 1 = \beta + n > \gamma) \Rightarrow (nad + bd = 0) \Rightarrow (na + b = 0) \Rightarrow (n = -b/a)$

Let $n = \max_{\mathbb{Z}} \{\gamma - \alpha + 1, \gamma - \beta, -b/a\}$. Replace Y by $a_n x^n + \cdots + a_0$ where the a_i are indeterminate, and reexamine the equation $AY' + BY = C$.

Each power of x has a coefficient that is now linear in the a_i. Comparing powers yields a system of linear equations in the a_i whose coefficients are in C. If we have a solution, we obtain y; otherwise, no such y exists.

EXERCISE: Apply this to

$$y' + \frac{2}{x} y = \frac{2x^2 + 4x + 1}{x^2(x+1)^2} \in \mathbb{Q}(x).$$

For sanity's sake, use Maple (or some equivalent CAS). Show that the solution is

$$y = \frac{2x^2 + (1 + a_0)x + a_0}{x^2(x+1)}$$

for any $a_0 \in \mathbb{Q}$.

2. We proceed to the second example. Let $f, g \in \mathbb{Q}(x)$ so that $y'' + fy' + gy = 0$. We want to determine if this has a solution in $\mathbb{Q}(x)$, and to find one if so.

We will take the same approach as before, using the same notation. Let

$$f = \frac{a}{p^\alpha} + \widetilde{f},\ g = \frac{b}{p^\beta} + \widetilde{g},\ y = \frac{a}{p^n} + \widetilde{y}.$$

Let $\overline{f}$ denote the remainder of f after division by p. Then

$$y' = -\frac{n\overline{qp'}}{p^{n+1}} + \cdots,\ y'' = \frac{n(n+1)\overline{(p')^2 q}}{p^{n+1}} + \cdots,$$

where we have exhibited the "leading terms". Plug this into the equation we wish to solve, and we have

$$\left(\frac{n(n+1)\overline{(p')^2 q}}{p^{n+2}} + \cdots\right) + \left(-\frac{n\overline{aqp'}}{p^{n+1+\alpha}} + \cdots\right) + \left(\frac{\overline{bq}}{p^{\beta+n}} + \cdots\right) = 0.$$

In order to have the cancellation, we must have the following four possibilities:

(a) $\alpha = 1, \beta \leq 1 :\ n + 2 = n + 1 + \alpha > n + \beta$

(b) $\alpha = 0, \beta = 2 :\ n + 2 = n + \beta > n + 1 + \alpha$

(c) $\alpha > 1, \beta > 2: \; n+1+\alpha = n+\beta > n+2$
(d) $\alpha = 1, \beta = 2: \; n+2 = n+1+\alpha = n+\beta$

EXAMPLE: For the fourth possibility, we have

$$n(n+1)\overline{(p')^2 q} - n\overline{aqp'} + \overline{bq} = 0.$$

Since $q \neq 0$ we have

$$n(n+1)\overline{(p')^2} - n\overline{ap'} + \overline{b} = 0.$$

This gives us an equation for n, for which there are two possibilities. We get similar equations for the other three possibilities (a), (b) and (c).

EXAMPLE: Let

$$y'' + \frac{6}{x}y' + \frac{6}{x^2}y = 0.$$

Observe that when we choose $p = x$, $a = 6$, $\alpha = 1$, $b = 6$, and $\beta = 2$. Then

$$\begin{aligned} n(n+1)\overline{(p')^2} - n\overline{ap'} + \overline{b} &= 0 \\ n(n+1) - 6n + 6 &= 0 \\ n^2 - 5n + 6 &= 0 \\ (n-2)(n-3) &= 0 \end{aligned}$$

so $n = 2$ or $n = 3$.

In fact, $y = 1/x^2$ and $y = 1/x^3$ are solutions.

We return to the general equation. Find the maximum integer root n_i. Set

$$y = \frac{Y}{\prod_i p_i^{n_i}}.$$

Plug in, clear the denominators again, and we have the polynomial equation

$$AY'' + BY' + CY = 0$$

with

$$A = ax^\alpha + \cdots, \; B = bx^\beta + \cdots, \; C = cx^\gamma + \cdots, \; Y = dx^n + \cdots.$$

Again we have four cases, each giving us a polynomial equation that n must satisfy. Let n be the largest integer root, as before, and substitute $y = a_n x^n + \cdots + a_0$. Then solve for the a_i.

Note that the linear system will be homogeneous. Thus, a basis of solutions $(a_n, \ldots, a_0)$ gives a basis of solutions of $y' + fy' + gy = 0$.