Notes for MA591U, Spring 2001 (Symbolic Computation)

Liouville's Theorem (Extensions of Differential Fields)

Let k be a field. Let $P(x)$ be an irreducible polynomial over k. There exists a field K so that:

(i) K constains k and a root of $P(x)$;

(ii) any subfield $\mathbb{F} \subset K$ so that $k \subset F$ and a root of P is in $\mathbb{F}$ has $\mathbb{F} = K$.

NOTE:

(1) $K = k[x]/\langle P(x)\rangle$.

(2) K is unique up to k-isomorphism; that is, if K and $\widetilde{K}$ are two such fields, then $\exists\varphi\; K \to \widetilde{K}$ such that φ is bijective and $\varphi|_k = \iota$. (For a proof, see Lang's *Algebra*.)

LEMMA: Let $\mathbb{Q} \subset \mathbb{F} \subset \mathbb{E}$ be fields with $\alpha \in \mathbb{E}$ and α algebraic over $\mathbb{F}$. If $\mathbb{F}$ is a differential field, there is a unique way to make $\mathbb{F}(\alpha)$ into a differential field, extending the derivation on $\mathbb{F}$.

PROOF:

Let $P(x) = a_n x^n + \cdots + a_0$ be a monic polynomial of smallest degree such that $P(\alpha) = 0$ and every $a_i \in \mathbb{F}$.

Assume first that such an extension of the derivation exists, and denote it by $(')$. We have

$$\begin{aligned} 0 = (P(\alpha))' &= (a_n\alpha^n + \cdots + a_0)' \\ &= (a_n'\alpha^n + a_{n-1}'\alpha^{n-1} + \cdots + a_0') + (na_n\alpha^{n-1}\alpha' + (n-1)a_{n-1}\alpha^{n-2}\alpha' + \cdots + a_1\alpha') \\ &= P_1(\alpha) + \tfrac{dP}{dx}(\alpha)\cdot\alpha'. \end{aligned}$$

Note that $\frac{dP}{dx}(\alpha) \neq 0$ since $\deg\frac{dP}{dx} < \deg P$. So

$$\alpha' = -\frac{P_1(\alpha)}{\frac{dP}{dx}(\alpha)}.$$

Hence the derivation in $\mathbb{F}$ determines α' uniquely.

So let $\beta \in \mathbb{F}(\alpha)$ be arbitrary, but fixed. It is of the form

$$\beta = b_n\alpha^m + \cdots b_1\alpha + b_0$$

with the $b_i \in \mathbb{F}$. Then

$$\beta' = (b_m'\alpha^m + \cdots + b_1'\alpha + b_0') + (mb_m\alpha^{m-1} + \cdots b_1)\alpha'.$$

Hence the derivation uniquely determines β' as well, and we have shown the uniqueness.

We do not show the existence; the idea is to set

$$\alpha' = \frac{-P_1(\alpha)}{\frac{dP}{dx}(\alpha)}.$$

This yields the desired derivation on $\mathbb{F}(\alpha)$. See Rosenlicht's paper for details.

DEFINITION: A differential field $\mathbb{E}$ is an **elementary extension** of some subfield $\mathbb{F}$ if one can write $\mathbb{E} = \mathbb{F}(t_1, \ldots, t_m)$ where, for each $i \in \{1, \ldots, m\}$, either

(a) $\exists u_i \in \mathbb{F}(t_1, \ldots, t_{i-1})$ so that $t_i' u_i = u_i'$ (that is, $t_i =$"$\ln u_i$"), or

(b) $\exists u_i \in \mathbb{F}(t_1, \ldots, t_{i-1})$ so that $t_i' = u_i' \cdot t_i$ (that is, $t_i =$"e^{u_i}"), or

(c) t_i is algebraic over $\mathbb{F}(t_1, \ldots, t_{i-1})$.

An element v of some field $\mathbb{K} \supset \mathbb{F}$ is **elementary over** $\mathbb{F}$ if it belongs to an elementary extension of $\mathbb{F}$.

EXAMPLE:

$$\sqrt{\ln(\ln(e^x + x))}$$

is elementary over $\mathbb{Q}(x)$, for one finds it in the field

$$\mathbb{F} = \mathbb{Q}(t_1 = x, t_2 = e^x, t_3 = \ln t_2, \ln t_3, \sqrt{t_4}).$$

EXAMPLE: e^{x^2} is elementary over $\mathbb{Q}(x)$.

THEOREM: *(Liouville)* There is no elementary extension of $\mathbb{Q}(x)$ containing an element y such that $y' = e^{x^2}$.

THEOREM: *(Liouville, 1835)* Let $\mathbb{F}$ be a differential field with $\mathbb{F} \supset \mathbb{Q}$. Let $\alpha \in \mathbb{F}$. If $y' = \alpha$ has a solution in an elementary extension of $\mathbb{F}$ having the same constants as $\mathbb{F}$, then there exist constants $c_1, \ldots, c_n \in \mathbb{F}$ and elements $v, u_1, \ldots u_n \in \mathbb{F}$ such that

$$\alpha = v' + \sum_{i=1}^{n} c_i \frac{u_i'}{u_i};$$

that is,

$$\int \alpha = v + \sum_{i=1}^{n} c_i \ln u_i.$$

REMARKS:

(1) If the constant field of $\mathbb{F}$ is algebraically closed, we do not need the hypothesis of "having the same constants."

(2) Otherwise, we need the assumption. For example, suppose $\mathbb{F} = \mathbb{R}(x)$. Then

$$\int \frac{dx}{x^2+1} = \frac{1}{2i}\ln(x-i) - \frac{1}{2i}\ln(x+i).$$

We claim that one cannot write the derivate of this expression in the form

$$\frac{1}{x^2+1} = v' + \sum_i c_i \frac{u_i'}{u_i}$$

for any $v, u_i \in \mathbb{R}(x)$ and for any $c_i \in \mathbb{R}$. The reason for this is as follows. Suppose $v = h + \sum\sum \frac{p_{ij}}{(q_i)^j}$ where the q_i are irreducible, and $\deg p_{ij} < \deg q_i$. Then

$$v' = h' + \sum\sum\left(\frac{p_{ij}'}{(q_i)^j} - \frac{jp_{ij}q_i'}{(q_i)^{j+1}}\right).$$

Now, if x^2+1 occurs in one denominator of v', then $(x^2+1)^2$ must also occur in another denominator (need to check that there is no cancellation). Assume that $u_i = \prod_j q_j^{n_{ij}}$ with q_j irreducible and $n_{ij} \in \mathbb{Z}$. Using logarithmic identity,

$$\frac{u_i'}{u_i} = \sum n_{ij}\frac{q_j'}{q_j}.$$

We can assume that the u_i are irreducible polynomials. Then

$$\frac{1}{x^2+1} = v' + \sum c_i \frac{u_i'}{u_i}.$$

If x^2+1 occurs in the denominator of v, then there is some $m \in \mathbb{N}$ such that $(x^2+1)^m$ occurs in the denominator of v' (we saw this above). Then there is some i so that $u_i = x^2+1$, and

$$\frac{u_i'}{u_i} = \frac{2x}{x^2+1}$$

and x^2+1 can *not* occur in a higher power (irreducibility). So x^2+1 cannot occur in the denominator of v, since it could not cancel; while it *could* occur as some u_i. In this case,

$$\frac{1}{x^2+1} = \cdots + c\frac{2x}{x^2+1} + \cdots.$$

By the uniqueness of partial fractions, we obtain a contradiction.

(3) If the constants of $\mathbb{F}$ are $\mathbb{R}$, then $y' = \alpha$ has an elementary integral if, and only if,

$$\int \alpha = v + \sum c_i \ln u_i + \sum d_i \arctan w_i.$$