# Notes for MA591U, Spring 2001
# (Symbolic Computation)

## Integration of Rational Functions

Suppose $f/g \in \mathbb{Q}(x)$. We want to evaluage $\int f/g$.

(1) We will show that:

$$\int \frac{f}{g} = R(x) + \sum_{i=1}^{m} c_i \ln(x - \alpha_i) \text{ where } R(x) \in \mathbb{C}(x),\ \alpha_i \in \mathbb{C};$$

(2) We will find the smallest extension $k \supset \mathbb{Q}$ so that

$$\int \frac{f}{g} = v_0 + \sum_{i=1}^{m} c_i \ln v_i \text{ where } v_0 \in k(x),\ v_i \in k[x],\ c_i \in k.$$

(3) We want to find the $v_i$ *quickly*.

**EXAMPLE:**

$$\int \frac{2x}{x^2+1} dx = \ln(x-i) + \ln(x+i) = \ln(x^2+1).$$

The key to the method we typically teach in calculus for the integration of rational functions is partial fraction decomposition (`parfrac` in Maple). Given $f/g \in k(x)$, there exist irreducible, monic polynomials $p_1, \ldots, p_m$ so that

$$\frac{f}{g} = h(x) + \sum_{i=1}^{m} \sum_{j=1}^{n_i} \frac{q_{ij}}{p_i}$$

where $h, q_{ij} \in k[x]$ and $\deg q_{ij} < \deg p_i$. This decomposition is, furthermore, unique. (For a proof, see Lang's *Algebra*.)

**EXAMPLE:** Consider

$$\frac{f}{g} = \frac{2x^3 - x^2 + 2x + 1}{x^4 + 2x^2 + 1} \in \mathbb{C}(x).$$

The decomposition is

$$\frac{f}{g} = \frac{-\frac{1}{2}}{(x+i)^2} + \frac{1}{x+i} + \frac{-\frac{1}{2}}{(x-i)^2} + \frac{1}{x-i}$$

so

$$\begin{aligned} \int \frac{f}{g} &= -\tfrac{1}{2} \int \tfrac{dx}{(x+i)^2} + \int \tfrac{dx}{(x+i)} - \tfrac{1}{2} \int \tfrac{dx}{(x-i)^2} + \int \tfrac{dx}{x-i} \\ &= \tfrac{1}{2(x+i)} + \ln(x+i) + \tfrac{1}{2(x-i)} + \ln(x-i) \\ &= \tfrac{x}{x^2+1} + \ln(x^2+1). \end{aligned}$$

In general,

$$\int \frac{f}{g} = \int h + \int \sum \sum \frac{q_{ij}}{(x-\alpha_i)^j} = H + \sum_{i=1}^{m} \sum_{j=2}^{n_i} \frac{q_{ij}}{(x-\alpha_i)^{j-1}} + \sum q_{i1} \ln(x-\alpha_i) = u_0 + \sum c_i \ln u_0$$

where $u_0 \in \mathbb{C}(x)$, $u_i \in \mathbb{C}[x]$.

The trouble with this method is that factoring introduces many things that *we don't really need.* In the example above, partial fraction decomposition introduced the algebraic extension $i$, but we did not need it in the final solution. Another example of this might be

$$\int \frac{nx^{n-1}}{x^n - 1} = \sum_{j=1}^{n} \ln\left(x - e^{j \cdot \frac{2\pi i}{n}}\right) = \ln(x^n - 1).$$

We would like an algorithm that uses as few algebraic extensions as necessary, and introduces them as late as possible.

In 1872, Hermite made the first development in this regard: we can find $u_0$ without factoring, and without going to extension fields.

(1) The first step is to write $g$ in squarefree factorization: $g = \prod_{i=1}^{m} r_i^i$. From the last example, $x^4+2x^2+1$, it is straightforward to show that $x^4 + 2x^2 + 1 = 1 \cdot (x^2+1)^2$.

(2) The second step is to write

$$\frac{f}{g} = r + \frac{b_1}{r_1} + \cdots + \frac{b_m}{r_m^m}$$

where $r, b_i \in \mathbb{Q}[x]$ (the $r_i$ come from the squarefree factorization of step (1)). We obtain $r$ by dividing $f$ and $g$. To obtain the $b_i$, we use the fact that $\gcd(r_i, r_j) = 1$ when $i \neq j$. Then there exist $a_1, b_1$ such that

$$a_1 r_1 + b_1 \prod_{i=2}^{m} r_i^i = 1$$

so that

$$\frac{1}{\prod_{i=1}^{m} r_i^i} = \frac{b_1}{r_1} + \frac{a_1}{\prod_{i=2}^{m} r_i^i} = \frac{b_1}{r_1} + \frac{b_2}{r_2^2} + \frac{a_2}{\prod_{i=3}^{m} r_i^i} = \cdots = \sum_{i=1}^{m} \frac{b_i}{r_i^i}.$$

To find $\int f/g$, it is thus enough to find $\int b_i/r_i$.

Hermite's idea showed how to reduce to the case where the exponent in the denominator is 1: find $C, D$ such that

$$Cr + Dr' = 1$$

(we can do this since $\gcd(r, r') = 1$) and then

$$\int \frac{b}{r^i} = \int \left(\frac{Crb}{r^i} + \frac{Dr'b}{r^i}\right) = \int \frac{Cb}{r^{i-1}} + \int Db\frac{r'}{r^i}.$$

The denominator of the integrand in the first term has a smaller degree than before, and the second integral we can integrate by parts. Let $u = Db$ and $dv = r'/r^i$. We obtain

$$uv - \int vdu = \frac{Db}{(1-i)r^{i-1}} - \int \frac{(Db)'}{(1-i)R^{i-1}}.$$

Observe that, again, the denominator of the new integrand has smaller degree than before. Hermite thus showed

$$\int \frac{f}{g} = H + \int \frac{b}{r}$$

where $H \in \mathbb{Q}(x)$, $\deg b < \deg r$, and $r$ is squarefree. We see that we can do all this without factoring, and without leaving $\mathbb{Q}(x)$.

We have now reduced the problem to

$$\int \frac{f}{g}$$

where $\deg f < \deg g$ and $g$ is squarefree. Rothstein and Trager independently solved this problem in the 1970s by showing the following.

**THEOREM:**
Let $R(y) = \text{Res}_x(f - yg', g)$ and let $c_1, \ldots, c_t$ be the roots of $R(y)$ in $\mathbb{C}$. Let $u_i = \gcd(f - c_i g', g)$. Then

$$(1)\ \int \frac{f}{g} = \sum_{i=1}^{t} c_i \ln u_i.$$

$$(2)\ \left(k \supset \mathbb{Q} \text{ and } \int \frac{f}{g} = \sum d_i \ln w_i\right) \Rightarrow \left[(c_i \in k) \Rightarrow (d_j \in k \text{ and } w_i \in k(x))\right].$$

**PROOF:**
We know, from the uniqueness of partial fractions, that $\exists \mathbb{F} \supset \mathbb{Q}$ so that

$$\exists d_i \in \mathbb{F}, \exists w_i \in \mathbb{F}(x)\ \left(\int \frac{f}{g} = \sum d_i \ln w_i\right).$$

What we will show is that inside of any such $\mathbb{F}$, we can manipulate $d_i$ and $w_i$ until we obtain

$$\int \frac{f}{g} = \sum c_i \ln u_i$$

with $c_i$ and $u_i$ as in (1) above.

Observe the following:

(1) $\ln(a/b) = \ln a - \ln b$, so we can assume that all the $w_i \in \mathbb{F}[x]$.

(2) $c\ln(pq) + d\ln(pr) = (c+d)\ln p + c\ln q + d\ln r$. Hence we can assume that the $w_i$ are relatively prime. (If $w_i$ and $w_j$ have a common factor $p$, factor it into its own logarithm as shown.)

(3) We can assume the $w_i$ are squarefree.

(4) We can assume the $c_i$ are distinct.

At this point, we claim that

$$\int \frac{f}{g} = \sum c_i \ln u_i$$

where the $c_i$ and $u_i$ are as above. For suppose $\sum d_i \ln w_i$ satisfies (1)-(4). Thus

$$\frac{f}{g} = \sum d_i w_i'.$$

Since the $w_i$ are relatively prime and squarefree, the uniqueness of partial fractions implies that none of the terms cancel. Hence $g = \prod w_i$. Let $W_I = \prod_{j \neq i} w_j$. Then

$$\frac{f}{g} = \frac{\sum d_i w_i' W_i}{\prod w_i}$$

and $g' = \sum w_i' W_i$. Now,

$$w_k = \gcd(0, w_k) = \gcd\left(f - \sum d_i w_i' W_i, w_k\right).$$

When $k \neq i$, $w_k | W_i$, so

$$w_k = \gcd\left(f - d_k w_k' W_k, w_k\right) = \gcd\left(f - d_k \sum w_i' W_i, w_k\right) = \gcd\left(f - d_k g', w_k\right).$$

When $l \neq k$, consider that

$$\begin{aligned} \gcd(f - d_k g', w_l) &= \gcd\left(\sum d_i w_i' W_i - d_k \sum w_i' W_i, w_l\right) = \gcd\left(d_l w_l' W_l - d_k w_l' W_l, w_l\right) \\ &= \gcd((d_l - d_k) w_l' W_l, w_l) = 1 \end{aligned}$$

(since $w_l$ squarefree implies $\gcd(w_l', w_l) = 1$, $w_l$ relatively prime implies $\gcd(w_l, W_l)$, and $d_l, d_k \in \mathbb{F}$). So

$$w_k = \gcd(f - d_k g', w_k) = \gcd\left(f - d_k g', \prod w_i\right) = \gcd(f - d_k g', g).$$

Hence the $d_i$ appear as roots of $R(y)$, and the $w_i$ are as given above.

Now we need to show that there are no other roots. Let $c$ be a root of $R(y)$ in some field $\widetilde{\mathbb{F}} \supset \mathbb{Q}$. Let $H = \gcd(f - cg', g)$. Let $h$ be an irreducible factor of $H$. Since $H|g$ and $h$ is irreducible, we have $h|g$. As $g = \prod w_i$ and the $w_i$ are relatively prime, $\exists j\ (h|w_i)$. Since $h|(f - cg')$, we have

$$\left[h| \left(\sum d_i w_i' W_i - c \sum w_i' W_i\right)\right] \Rightarrow \left[h| \left(d_j w_j' W_j - c w_j' W_j\right)\right] \Rightarrow (d_j = c).$$

**EXAMPLE:** Observe that

$$\int \frac{1}{x^3 + x}$$

has $f = 1$ and $g = x^3 + x$. Also, $\gcd(g, g') = 1$, so $g$ is squarefree. We now want

$$R(y) = \text{Res}_x(1 - y(3x^2 + 1), x^3 + x) = \cdots = (2y + 1)^2 (y - 1).$$

Here $c_1 = -1/2$ and $c_2 = 1$. Then

$$u_1 = \gcd\left(1 + \frac{1}{2}(3x^2 + 1), x^3 + x\right) = \gcd\left(\frac{3}{2}x^2 + \frac{3}{2}, x^3 + x\right) = x^2 + 1$$

$$u_2 = \gcd(1 - (3x^2 + 1), x^3 + x) = \gcd(-3x^2, x^3 + x) = x$$

so

$$\int \frac{1}{x^3 + x} = -\frac{1}{2}\ln(x^2 + 1) + \ln x.$$

This was of course an easy integral to compute, as there were no extensions that we needed to make to $\mathbb{Q}$, but this is not necessarily the norm.